What's an Assignment Like You Doing in a Course Like This?: Writing to Learn Mathematics[1]

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Over the past generation or two, many college mathematics professors have been pressured to "service" an increasing number of poorly prepared students in courses such as calculus and statistics. In response, we have created memory-based courses, driven by efficient means of testing, in which success is defined in terms of calculational skill. We may rationalize that accuracy in computation implies a previous mastery of concepts, but we all know better.

Meanwhile, technological developments (calculators and computers) have rendered obsolete many of the very techniques we emphasize in our courses, especially for the students we "serve," the overwhelming majority of whom will not pursue careers in mathematics per se. Conceptual mastery may have been needed in the past to compute accurately, but that need has been significantly reduced by the sophistication of the technology now available to students.

Needing a new way to re-emphasize conceptualization in the mathematics curriculum, more of us have become willing to consider the pedagogical efficacy of writing assignments, which force students to (re)articulate concepts before pushing the

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buttons. This new hope assumes that *thought* and *expression of thought* are so closely interrelated that to require the latter will engender the former.

Immediately problems arise: What kinds of writing assignments would produce the desired effects? What kinds of magisterial responses would be called for? Would the mathematics teacher suddenly be forced to become a part-time writing teacher? Would the added time burden (of reading and responding) be bearable and cost effective?

The 1986 calculus conference at Tulane University [**1**] strongly recommended making writing assignments a regular part of calculus courses. At Duke University we have embarked on an experimental course as a first step in developing a new calculus curriculum. We began with (1) a mathematics professor interested in investigating the possibilities and (2) a new methodology for analyzing and teaching writing, compact enough to be imported into the mathematics classroom and effective enough to make it worth the import. With this paper we wish to share some of the problems we faced, the first results of our experiment, and selected principles of our methodology.

A Course on Calculus, Computers, and Words

In our two-semester course entitled *Introductory Calculus with Digital Computation*, freshmen discover they are faced with the new and mysterious task of *writing mathematics*. The content of the course includes both the standard first-year calculus syllabus and a not-so-standard computer laboratory component. The computer-related material requires that the class meet an extra hour each week (the lab period), and that students do lab assignments on their own time in two-person teams. They write weekly reports on their lab experiences to demonstrate their comprehension of the concepts involved and their process of achieving that comprehension.

These weekly lab reports typically include data, tabulations, graphs, and 1–3 pages of expository writing. We evaluate the students primarily by three open-book, take-home tests per semester, each with three substantial problems whose solutions are written in essay form. The students are *ordered* to collaborate, to learn as much as they can from each other, and to write what they learn in their own words; thus, there is no opportunity to "cheat." The semester final examination takes place in a conventional setting (three hours in a classroom, no collaboration, open-book); it has two somewhat less substantial problems to be written out in full, plus an essay question on the meaning and importance of one of the major theorems. In addition, we require regular homework (conventional exercises, no writing) and "mastery" tests of basic computational skills (open book, no writing, taken until a 95% score is achieved).

The Dissociation of Words and Numbers

The very idea of writing in a mathematics course is foreign to most students. Witness the opening of one of the early student reports:

> Once upon a time, in an Engineering Building, far, far away,... there was a computer cluster. To this cluster journeyed two dutiful slaves of Calculus. These weary travelers had journeyed far, from the very reaches of East Campus, in order to ask their simple questions and calculations of the Great MicroCalc Program. Sent

on their quest by the High Wizard Smith, the only directions they were given were...

a) to study symmetric difference quotient approximations to values of the derivative, with the object of finding an appropriate x to get good approximations for reasonable functions, and
b) to use the selected x to construct a derivative tabulator that is independent of Calculus formulas.

This folkloric/gothic/Oz metaphor, maintained throughout the report, solved for these two students the befuddling puzzle of having to introduce "writing" into their math homework. To them, "writing" was something learned in English classes, something Hemingway, Fitzgerald, and Hawthorne did. It has something to do with "style," but not necessarily with "thought." If "writing" has come to math, that must mean that math must now be done with imagination and "style." Hence the metaphor.

"Thought" (they believe) is the sort of thing encountered more often in courses devoted directly to the subject—philosophy, history, government, psychology, and the like. Mathematics is in yet another category altogether. While it is clear to them that you have to work hard mentally to solve the problems, you do not necessarily have to have "thoughts." Curiously, there is a connection between the misconceptions our students suffer about the relationships of thought to mathematics and thought to writing: In both cases, thought seems to be something anterior to the other activity. According to students, you "think" first (mechanically), then you do the math problem; you "think" first (conceptually), then you "reduce your thought" to writing. In order for students to benefit more fully from their training in calculus, they must come to understand that they are engaging in a process of thought, in a new mode of thought. Forcing them to *write* about what they are doing will in turn force them to *think*, to conceptualize about what they are doing. At the same time, we need to demonstrate to them the inextricable intertwining of thought and writing —of thought and expression of thought. Duke's new programs are currently making this double attempt: to teach mathematics better through writing, and to teach writing better through mathematics (and chemistry and philosophy and history...).

We should not blame our students for these misconceptions. Why should they think otherwise, when throughout their former training the subjects mathematics and English have been so rigidly segregated? The subjects are taught by different teachers; the books have a different look to them; even the all-perceptive (they think) college aptitude tests must be divided into "verbal" and "mathematical." Perhaps most convincing of all, a student is allowed to be good in one and relatively poor in the other and still remain in everyone's eyes a good student, intelligent, even stunningly bright. In fact, it is a relative rarity to find the student who is a genuine double threat, outstanding with both words and numbers.

Students tend to infer from this that numbers and words signify differently. (We use "numbers" generically to represent arbitrary mathematical objects, symbols, and constructs.) To students, numbers always imply truth, while words often produce mere concepts; not only do numbers *have* boundaries, but essentially they *are* boundaries. It might seem that words have individual boundaries (or what's a dictionary for?); but words often have several different definitions, and the combination of words (into sentences, paragraphs, essays) raises so many possible permutations of interpretation that it becomes one of the major objectives of expository prose to establish boundaries.

Words strain,
Crack and sometimes break, under the burden
Under the tension, slip, slide, perish,
Decay with imprecision, will not stay in place,
Will not stay still. [**2**]

In many senses, numbers "stay still." Our mathematical symbolism has evolved to obviate the difficulties of multiple interpretation inherent in verbal texts. As recent literary theory argues [**3**], a verbal text has as many interpretations as there are readers of the text; the text does not exist by itself, as an indelible expression of authorial intent, but only as a product of the intersection of text and perceiver of text. Without entering into the debate, we simply note that no one is arguing analogously for numbers. Virtually the entire community of number users and perceivers, acting like one collective author, agrees on the meaning and function of mathematical symbols. By having writing assigned in their calculus courses, then, students may indeed be puzzled as to how to apply slippery, sliding, constantly re-interpretable words to a subject previously infused with the truth and unchanging exactitude of numbers. The numbers and the formulae have become for them the thought itself, no longer the symbol of thought. We are asking them to abandon that dissociation.

The model of mathematical prose most available to students is their textbook; however, the writing found there is often less than effective, and the students often avoid reading it. We trace the blame for this to generations of combined efforts from two quarters: On the one hand, authors and publishers produce textbooks that do not have to be read before doing the exercises; on the other hand, teachers acquiesce by agreeing that this is the way mathematics ought to be taught. What prose there is has tended to be introductory, apologetic, and self-justifying. It implies that the real importance lies not in the students' ability to conceptualize, but rather in their ability to compute. Teachers tend to underscore this by their rapt attention to correctness, completeness, and procedure. Students comply with the grand scheme by establishing as their local goal the correct completion of a given assignment and as their global goal receiving their desired grade in the course. For most, once it's over, it's over.

Common Problems with Student Writing in Mathematics

Asked to write in English what they are doing with mathematics, our students tend to settle for narrating (not explaining) the steps they take in solving a mathematical exercise. This leads (predictably) to a litany of problems that we summarize here, using examples from the earliest lab reports of the Fall 1987 semester.

We chose this source of examples for two reasons. First, these are the students' *first* (graded) writing efforts. (The first take-home test comes after at least three labs.) As we will explain later, the instructor's responses to the earliest attempts at writing are crucial for achieving success. Second, these assignments were given when the students had completed less than two weeks of their University Writing Course. We also explain later the operative principles of the writing course; for now, we merely observe that the students were simultaneously learning these principles and learning to apply them in calculus.

Our written instructions for each lab included sections on Purpose, Preparation (usually background reading from a supplementary text), Lab Project (an outline of the explorations to be carried out, usually with questions to be answered, some of which were open-ended), and Lab Report. Early versions of the Lab Report section included instructions on the importance of and proper handling of *data* (a concept foreign to most students' experiences with mathematics). Every Lab Report assignment included some version of the following statement:

> Your written report should describe the *process* by which you studied [whatever topic]: the decisions you made and why you made them, mistakes you made and how you corrected them, observations you made and what you learned from them. It should also state your *conclusions* (including answers to specific questions raised) and the *evidence* supporting those conclusions.

The examples here and in subsequent sections should be read in the context of this assignment, bearing in mind that the authors were early-stage freshmen for whom much of the assignment was a mystery.

Since the primary purpose of the first lab was to gain familiarity with the hardware (IBM PC's and compatibles) and software (MicroCalc [**4**]), the mathematical content was modest: Tabulate and graph three particular functions, answer some questions about approximate locations of their zeros, and describe the relationships among the functions and their zeros. (Two of the functions were the first and second derivatives of the other.) All the examples in this section are from first drafts of this first lab report.

Difficulty in finding conceptual rather than factual content: "What is there to write?" The first problem students had with prose was finding anything to say. Their preferred model was the typical math homework paper, which tends to contain little more than a list of answers, perhaps supported by a sanitized verbal version of their calculations. Thus we found:

> "The value of $f(-2)$ is as follows:
> $$f(-2) = -33."$$

If this signifies anything beyond what the symbols alone would have conveyed, perhaps it reveals the student's inability even to interpret the equation verbally: "The value of f at -2 is -33."

Failure to connect narrative with data or to support conclusions with evidence.

> "We located a root of $f(x)$ at $x = 18.74$."

The location of the zero happened to be correct, but the computer-drawn graph this team submitted was of the wrong function; there was no indication of any calculation to support an answer of this accuracy. Having reached the concluding response of "18.74," it seemed unnecessary for them to explain or exemplify the process behind the word "located."

The sensed connection not made explicit: "I see it, but I can't say it." The second clause of the following sentence asserts two things about H.

> "From our previous study of calculus, we determined that $G(x)$ is a first derivative of $F(x)$, and $H(x)$ is a second derivative of $F(x)$ and a first derivative of $G(x)$."

The student did not articulate the connection between "derivative of a derivative" and "second derivative," leaving open the possibility that she did not fully understand the obvious connection.

Denial, suppression, minimization of mistakes. Students were frequently reminded that "mistakes are the best teachers" and advised that they would receive credit for analyzing accurately how their mistakes had been made. However, the message they have internalized over many years is, "mistakes are what you get points taken off for." Here is an example that reveals such authorial agony:

> Due to brief misinterpretation of questions two and three, only one x value was sought and found. Only after leaving the computer facility was it discovered that two or three values were required. Hence, some of the value tables were constructed using a simple 'home-grown' program on an Apple II and do not contain as accurate a scale as the tables printed with MicroCalc.

Students commonly repaired perceived errors by scrambling instead of by rethinking. Their writing tended to reveal the scramble:

> Any difficulties in the lab occurred because some numbers were not quite accurate. The computer only carried out the figures to the sixth decimal place, and the students tried to make the results as accurate as possible by narrowing the range, a and b on the table of values while at the same time increasing the number of intervals, N to up to 500.

In fact, the exercise asked for "approximate" numerical results, for which a table of, say, 20 numbers would have been quite adequate.

A Source of Help: Duke's Writing Across the Curriculum Program

We have been discussing the problems that beset students when they are required to write in mathematics classes. What of the problems that will beset the instructors? Must they learn methods for helping students with their writing? Are they to add to their already substantial burden the new and uncomfortable task of teaching writing? Will the new requirements conflict with the main task at hand, teaching mathematics?

Duke University is building a Writing Across the Curriculum Program that supplies the kind of help a mathematics teacher might need. The major difference between the Duke program and others is its reliance on a new methodology for teaching and analyzing writing. This methodology has proven effective in doing consulting work with corporations, law firms, and governmental agencies; it has been effectively taught for six years at the Harvard Law School; and it is now in use in the undergraduate programs at Duke and the University of Chicago. The creators of this theory are Professors Joseph Williams (University of Chicago), Gregory Colomb (Georgia Institute of Technology), and George Gopen (Duke University). The concepts involved apply to all disciplines and can be applied by faculty from all parts of the university.

The methodology differs from all other strategies in the way it forsakes the more traditional perspective of "writer strategy" for the newer perspective of "reader expectation." "Writer strategy" asks "What can the writer think of to say next?" Such an approach probably grew out of the more immediate problem that afflicted the writing course instructor—how to fill several weeks with assignments engaging enough for students to be motivated to write anything at all. "Reader expectation" asks the more pertinent and lasting question, "Is the reader likely to come away from this prose with the precise thought(s) I intended to communicate?" Effective methods for finding answers to that question will help students to write better *every* time they write, whether it be in their writing class, in a math class, or in any of life's many rhetorical tasks.

"Reader expectation theory" was born of the linguistic discovery that readers expect certain components of the *substance* of prose (especially context, action, and emphatic material) to appear in certain well-defined places in the *structure* of prose [**5**], [**6**], [**7**]. Once consciously aware of these structural locations, a writer can know how to make rhetorical choices that maximize the probabilities that a reader will find in the prose precisely what the writer intended the reader to find.

Readers have what we call "reader energy" for the task of reading each different unit of discourse. (A unit of discourse is anything in prose that has a beginning and an end: a phrase, clause, sentence, paragraph, argument, article, book, etc.) Those energies function in a complex simultaneity: one reads a clause while reading its sentence, which is also part of the chapter.... Each of those energies is available for two major tasks: (1) for perceiving structure (how the unit of discourse hangs together); and (2) for perceiving substance (what the unit of discourse was intended to communicate). For the most part, the distribution of this energy is a zero-sum game: Whatever energy is devoted to one of these tasks is thereby not available for the other. For most expository prose, one almost could define bad writing as that which demands a disproportionate amount of reader energy for discovering structure. If a reader is spending most of the available reader energy trying to find out where the syntax of a sentence resolves itself or how this sentence is connected to the sentence that preceded it, that reader can have precious little left for considering what ideas the writer is trying to communicate. On the other hand, if the resolutions and connections appear exactly where the reader expects them to appear, then the reader can devote most of the available reader energy to perceiving the nature of the substantive thought.

Placing information in one structural location instead of another results in subtle *but remarkably significant effects. That subtlety requires that we treat in some detail* one example for which we know the authorial intentions (because she told us). Please bear with the non-mathematical example.

Compare these two sentences:

(a) What would be the employee reception accorded the introduction of such an agreement?
(b) How would the employees receive such an agreement?

Putting aside questions of "better" and "worse," we can probably agree that (b) is *easier* to read than (a). What makes that so? At first we might suggest that (b) is shorter than (a); it turns out, however, that the reduced length is a manifestation of improvement, not a cause of improvement. "Omit needless words" is helpful advice only to those who know already which words are needless.

Instead, we would do better to investigate exactly what is going on in the two sentences, what *actions* are taking place. When we seek out the possible action words of the first sentence, we find several candidates: "be," "reception," "accorded," "introduction," and "agreement." Intelligent readers can find good arguments for interpreting any of these as actions; the author of this intended to convey action in only two of them.

When we turn to the (b) sentence, our task is significantly simplified. It is clear to a majority of readers that "receive" is the one and only action happening in this sentence.

Why the great difference? Because readers *expect* to find the action of a sentence in the sentence's verb. That expectation leads us to perceive action in the verb slot unless that perception makes no sense. In sentence (a), "accorded" sounds like an action but makes no sense as an action. When that expectation is foiled, we have to look elsewhere in the sentence to find the action. Unfortunately, readers have no expectations concerning a secondary structural clue. All that remains is a set of highly interpretable semantic clues: "Reception"? "Introduction"? "Agreement"? Perhaps some concept not actually named by a word on the page? We are using our reader energy for hunting through the structure to find something that the writer could have pointed out to us easily by depositing it in the verb slot. Fulfilling the reader expectation (that the action will appear in the verb) greatly increases the probability that the reader will perceive what the writer intended the reader to perceive.

When something is badly written, more than cosmetic grace is at stake; communication itself may falter. As it turns out, the author of sentence (a) complained that (b) was an inaccurate revision of (a), since it omitted the concept of "introduction," which she had intended as a significant action. New solution: If "reception" and "introduction" are both actions, make them both into verbs. Here is what the author had intended to say:

> (c) How would the employees receive such a proposal if the Council introduced it at this time?

Note that this revision has forced an articulation of *who* is doing the introducing ("the Council") as well as a qualification ("at this time") that makes the action worth considering here.

Revision (c), according to the author, articulates clearly what her intentions had been. Revision (b) may have been brisker and easier to read than (a), but it failed to retain the author's intentions. The problem of the prose in (a) had not been merely a lack of grace, but rather a lack of clarity. We would argue that the two cannot be separated.

Reader expectation theory allows readers to identify a lack of clarity by perceiving a difficulty in structure. We may not know what is missing from the thought, but we can learn how to ask the right structural questions ("What *did* you intend the action of this to be?" "What verb would articulate that?") that will eventually turn up the answers if the author is present. These principles, then, do not give the instructor the power to revise students' prose effectively, but rather allow the instructor to help students revise their own prose. Only the author knows what the author intended.

The same kinds of discoveries concerning reader expectations have been made on the sentence level for the locations of context ("where am I coming from?") and of emphasis ("what is new and important here?"). Yet others have been discovered for the linking material between sentences, for the placement of points in paragraphs, and for the placement and development of thesis statements in complete essays. None of this material is strikingly new in and of itself; good writers, upon hearing the principles, will nod and say they "knew" that, although they had never heard it put quite that way before. The newness of the methodology lies in its having achieved two things: (1) Principles that have been mostly intuitive to this point are now objectified and made conscious for the writer, the better to be controlled and used; and (2) there is now a systematized language with which to speak of reader expectations, no matter what the nature or the field of the substantive material.

These principles have the potential to revolutionize writing instruction. They can be taught to teachers in workshops that last no more than 12–15 hours. They can be used immediately and with wide-ranging effectiveness. They can be taught to students who in turn can use them in evaluating each other's prose, so that peer commenting and grading of writing can become one of the most useful of teaching strategies. Most importantly, the principles allow an instructor to comment not simply on what a student has done "wrong" in a given paper, but rather on what ineffective rhetorical choices a student tends to make with great consistency. The student who puts the action elsewhere than in the verbs throughout one paragraph is highly likely to do so in other paragraphs. Teaching such a student about verb/action reader expectations will aid that student not only to revise the present paper effectively, but also to avoid that structural pitfall in all future writing tasks.

As the product of such revision is by no means merely cosmetic, so the process of that revision is by no means merely mechanical. In order to "fix" a sentence whose action does not appear in the verb slot, a writer has to ask the salient and substantive question "What is going on in this sentence?" The writing process, including this kind of revision process, does not merely lead back into the thinking process; it *is* a thinking process. Eventually the methodology transforms itself from a set of revision tactics to a set of invention procedures. Knowing how structures need to be built eventually leads the writer to recognize logical progressions while still engaged in the original writing process. The result is sometimes a quicker pace of writing, usually a greatly reduced need for revision, and almost always a clearer, more forceful product.

A note about what this methodology does *not* do: It does not propose a new set of rules which must be slavishly followed. It only makes a writer aware of the expectations that most readers have most of the time; the writer can then choose to fulfill those expectations or to foil them. Every one of these reader expectations can be violated to good effect. In fact, the greatest of stylists turn out to be the best violators. (This can be done only if the reader expectations are regularly fulfilled, so that the violation comes as a surprising exception.)

Knowledge of these reader expectations, therefore, should not be used to establish a new set of "rules" akin to our grammatical requirements for coherence (e.g., "use singular subjects only with singular verbs") or our grammatical conventions (e.g., "never split your infinitive"). We cannot and do not intend to argue that the action of sentences *must* be articulated by its verbs; sentences that have their actions elsewhere (or nowhere) abound in the published prose of all fields. Those sentences are not impossible to interpret; they are only less likely to be interpreted by a great many readers in the manner intended by the author. We speak, then, not of rules but of predictions of reader expectations. As a result, we do not list here all the major reader expectations; together they would take far more space to explicate than is here available. A full-length book on the subject is forthcoming.

Duke requires all freshmen to take one of our small (10–13 students) University Writing Courses in their first semester. They are taught the full sweep of the methodology, and they are given opportunities to teach it to each other through a series of peer evaluations. We now have an entire undergraduate student body able to talk the same language about language. Moreover, well over half of our faculty have attended the workshops. As a result, teachers and students in every department are now able to communicate directly, concisely, and effectively about the strengths and problems of students' prose.

The Methodology at Work: Solving Rhetorical Problems in Mathematics

We return now to examples from the earliest lab reports. The rhetorical problems we address here are manifestations of the student-centered problems described earlier, but they do not match up in one-to-one correspondence. The methodology as a whole will affect the student's thinking process as a whole, with the result that both kinds of problems will be substantially reduced.

The list-of-facts paragraph. Perhaps the single most common writing problem in the lab reports resulted from students merely listing their observations, in pseudo-paragraph form. That is, they would indent the first line (therefore "paragraph"), but fail to construct the connections between sentences that would create a coherent unit of discourse (therefore "pseudo").

Reader expectation theory offers several ways of combatting this problem. One can begin by checking the verbs in each sentence. If they consistently fail to state the action of the sentence, it is highly probable that the burden of making the decisions concerning both cohesion and coherence will have been shifted from the writer to the reader.

In the following example we have underscored each of the verbs.

> We observed that the function h is the derivative of the function g, which, in turn, is the derivative of the function f, therefore the function h is the second derivative of the function f. MicroCalc confirms these observations in the Derivatives unit of Quarter 1. Our second observation, concerning the f-g-h relation, was that the relative maxima and minima of an equation are equal to the x-values of the point where the derivative of that equation crosses the x-axis. MicroCalc confirmed this observation also, in the Extrema unit of Quarter 1. The final observation reveals the connection between the f-g-h relationships and questions 3, 4, 5 and 6 answered below.

Note how (with one exception) they all are verbs of observing (on our part), existence (on the part of our observations), or confirmation (on the part of MicroCalc). What does this paragraph tell us?—that the students observed certain things, and that MicroCalc confirmed their observations. It does not convince us that the students have *conceptualized* anything that they have managed to observe.

The purpose of assigning writing in this course was to force students to think about that which they might otherwise perform in only a mechanical way. The constant presence of weak verbs allows the mechanization to continue, the prose acting not as an explanation of thoughts (exposition) but merely as a recitation of facts (narration). Requiring the student to articulate the action of the sentence in the meaning of the verb will force the student to consider what actually is happening during the mechanized process. That, in turn, transforms what was only narrative prose into expository prose. For example, there is one verb in this example that actually does articulate the action of its sentence: "crosses." If the verbs in the surrounding sentences were made to articulate appropriate actions, then the student would be likely to recognize not merely the existence of the derivative crossing the x-axis, but also its significance.

The overpacked sentence with incredible connections.

> We made a table of values for x of $[0,20]$ and narrowed down the interval to $[15,20]$ as well as observing, from Table 1, that at $x=0$, $f(x)$ also equals 0.

In Lab 1, the function f was a polynomial with non-zero constant term; thus, the final assertion in this example is false. However, the connections in the sentence are deceptive, not actual, and thereby distract the reader from noticing that the conclusion is incorrect.

The author of this example could just as easily have continued to add phrases and clauses. No need to stop there: "... $f(x)$ also equals 0, as well as..., after which we..., which resulted in... ." It all would make perfect sense to the writer, who was proceeding linearly through time. However, it would become an increasingly annoying and confusing burden to the reader. How can we tell our students that they have packed too much into one sentence?

Again we can turn to reader expectation theory for help. Readers expect the single most important thing in the sentence—the thing that the writer intends the reader to emphasize—to appear at the end. We call this location the "stress position." Stress positions are created by syntactic closure. Hence every sentence has a stress position at its end, as the period brings it to a close. Colons and semicolons are strong enough acts of closure to create secondary stress positions in the middle of sentences. A sentence is too long when there is more than one viable candidate for any stress position.

In the above example, it is unclear just how many things the writer wishes us to emphasize. As a result, we also do not know precisely the relative weights and the explicit connections we should be observing. Do either of the following express more accurately the authorial intent?

> In making a table of values for x of $[0,20]$, we narrowed the interval to $[15,20]$.—[Reader's thought: What's the connection?]—From this table we observed that at $x=0$, $f(x)$ also equals 0.
>
> We chose to limit the table of values for x to $[0,20]$ because... . Similarly, we narrowed the interval to $[15,20]$ because... .—[What's the connection?]—We observed from the first table that at $x=0$, $f(x)$ also equals 0.

Given the author's prose, we now see that we are unable to perceive the authorial intent. Notice how much thought and information we lack to make the proper balances and connections. Insisting on one emphatic point per stress position will unearth (or stimulate) a great deal of student thought. It will also make it clearer to students when they yet have no point to make.

Lack of agency: Nobody did anything; things just happen; things just are.

> The function g is the derivative of the function f. The function h is a derivative of the function g. This is displayed on the table marked C. ...

Note how nobody *does* anything here. Students were asked to report what *they* saw and did, and yet they are nowhere to be found in the prose. Some will complain that they were carefully taught in high school not to write in the first person. Reader expectation theory demonstrates that the herring of first person usage is a red one. It matters not whether one ought or ought not to use the first person; it matters only whose story is being told. Readers firmly expect that the story of a clause or sentence is about whoever appears at its beginning. If a lab report must tell what students observed, then students ought to appear quite often at the beginning of clauses and sentences.

> We discovered that the function g is the derivative of the function f, because.... Using the same principle, we also perceived that the function h is a derivative of the function g. (See Table C.)

Combining the principles of action-in-verb and whose-story-up-front, students can lead themselves to discovering the emptiness of their own statements. We were surprised, for example, by the frequency with which students wrote that they "received" answers. That formula—"We received the answer of 3.742 for x"—reveals and sums up for us the central pedagogical problem in calculus courses.

In sentences written in the active mode, the person whose story it is also turns out to be the agent of the action. Students find they can avoid some of the consequences of making mistakes if they substitute other agencies for their own at the beginnings of sentences. For example (from Lab 2, devoted to turning points):

> The complexities of the graph of $f(x) = (x^4 + x - 1)/(x^3 + 2x^2 - 1)$ resulted in us not being able to correctly determine any turning points on the graph of that function.

Whose fault? "The complexities'."

Take a second look at an example we cited earlier:

> Any difficulties in the lab occurred because some numbers were not quite accurate. The computer only carried out the figures to the sixth decimal place, and the students tried to make the results as accurate as possible by narrowing the range, a and b on the table of values while at the same time increasing the number of intervals, N to up to 500.

In the first sentence, "difficulties" occurred. In the second, "the computer" fails badly to do its job, but "the students" come to the rescue ingeniously. If the students had been active earlier in the prose, they might have come to understand that for the approximate results required, a much shorter table (20 numbers) would have sufficed. By escaping responsibility here, they landed themselves with several times the labor.

The multi-topic paragraph. Here is yet another fact-filled, chronologically linear paragraph.

> From our previous study of calculus, we determined that $G(x)$ is a first derivative of $F(x)$, and $H(x)$ is a second derivative of $F(x)$ and a first derivative of $G(x)$. We confirmed our assumption by using the 'Derivative' command from MicroCalc. We took the first two derivatives of $F(x)$, and they were the same as $G(x)$ and $H(x)$, respectively. Next, we found the first derivative of $G(x)$, which resulted in the function $H(x)$. We also observed that each time $F(x)$ would reach an extreme point $G(x)$ would be equal to zero. This held true for $G(x)$ and $H(x)$. When $G(x)$ reached an extreme, $H(x)$ also equalled zero. We confirmed this by using the 'Extrema' command on MicroCalc. Using this command we found out the extreme

values for $F(x)$. Then, we plugged those x-values into $G(x)$, and $G(x)$ was found to be zero.

In addition to all the revision tactics suggested above, we can approach this example in terms of reader expectations of paragraph structure. Readers expect an expository or argumentative paragraph to be about one main point. That point should normally be expressed in a single sentence, from which or to which all other sentences flow. This paragraph violates reader expectations significantly:

1) There are too many points being made for the number of stress positions available;

2) No one sentence articulates the point that gives shape to the rest of the paragraph.

We have found, however, that students tend to have few problems with the concept of the *unity* of the paragraph. Indeed, they seem delighted to bring their momentary labors to closure by ending the topic and indenting the next line. How, then, can the above example of prose have been written as a paragraph? The answer we suggest to our own question we find most disturbing. To the student, the entire paragraph was indeed on one topic—the most important topic of all: "How much can I say about this problem so that I can get full credit for the answer?" All the sentences in the paragraph connect to that (unstated) point sentence.

We need to move our students away from being centrally concerned with producing enough material and direct them towards grasping and developing concepts. We can help accomplish this by getting them to understand the reader expectations of paragraph structure. They will be hard pressed to fulfill those expectations without entering into the kind of conceptualizing thought process that we wish them to experience.

Two Before-and-After Samples

We provide here two samples from the first lab reports that display several of the rhetorical problems described above. Each sample (a) is followed by responses from the instructor to the team of authors (b) and by the second submission of the equivalent text (c). In the section that follows we will describe a "peer response" exercise conducted in class and as homework. This exercise used these same lab reports, so each team had both peer and instructor comments to use in their revision. The assignment for this lab was sketched earlier, in our section on common (student-centered) problems.

1a. First draft:

> Due to brief misinterpretation of questions two and three, only one x value was sought and found. Only after leaving the computer facility was it discovered that two or three values were required. Hence, some of the value tables were constructed using a simple 'home-grown' program on an Apple II and do not contain as accurate a scale as the tables printed with MicroCalc.

This team has just barely admitted that they had made a mistake; they counterbalance that with an overly apologetic explanation of its repair. The authors actually had been most creative in repairing their error but were not confident enough to take appropriate credit. But they were also hiding the problem of having started the lab too close to the deadline to permit a second try.

1b. Instructor's comments:

> Whose story is this? (Nobody actually did anything.)
>
> Check the verbs. Do they express the action? (For example, the action "to misinterpret" appears in the nominalization "misinterpretation.")

The second version still has problems, but the improvement is evident. The deadline-crowding problem remains hidden; but the students have identified themselves as agents, and their actions are now expressed in the verbs. In particular, they have found that they *can* say "we solved our problem."

1c. Revised version:

> As we proceeded through the lab, we encountered one small problem: we misinterpreted questions two and three. To be exact, we sought and found only one x value as an answer. Only after leaving the computer facility did we discover that two or three values, and not simply one, existed. Hence, we solved our problem by constructing some of the value tables using a simple 'home-grown' program on an Apple II.

Our second example is a typical list-of-facts paragraph, featuring weak verbs, missing agencies, and absent logical developments or reasons.

2a. First draft:

> The relationship among the functions f, g, and h is one in which $g(x)$ and $h(x)$ are the first and second derivatives of $f(x)$, respectively. This means that $g(x)$ is the slope of the tangent line to $f(x)$ and $h(x)$ is the slope of the tangent line to $g(x)$. Therefore, whenever there is a turning point in $f(x)$, the graph of the derivative of $f(x)$, $g(x)$, will cross the x-axis. Likewise, whenever there is a turning point in $g(x)$, the graph of the derivative of $g(x)$, $h(x)$, will cross the x-axis.

2b. Instructor's comments:

> Whose story is this? [For reasons discussed earlier, almost every student had a problem initially with assuming the role of agent.]
>
> Check the verbs. Do they express the action?
>
> What is this about? Which is the point sentence? Is there more than one?
>
> What have you emphasized in each stress position?
>
> How are the sentences connected? Where is the logical development?

Again, many of the problems are attended to by the revision. Note especially that the authors have discovered that they had really been discussing two topics; as a result, they revised the one paragraph into two. The revision also contains a correct embedding of simple equations in a complete sentence, a difficult concept for most students.

2c. Revised version:

> Observing that the exponents of the first terms of the functions decrease by one between $f(x)$ and $g(x)$, and between $g(x)$ and $h(x)$, we took the derivatives of the

functions. We discovered that $f'(x) = g(x)$ and $g'(x) = h(x)$. We concluded that $g(x)$ is the first derivative of $f(x)$ and $h(x)$ is the second derivative of $f(x)$. We confirmed this by using the Derivatives unit of MicroCalc.

We noticed that, as a result of this relationship between the functions, whenever $f(x)$ changes direction, the graph of $g(x)$ crosses the x-axis. Likewise, whenever $g(x)$ has a turning point, the graph of $h(x)$ crosses the x-axis. This observation led us to be even more certain that $g(x)$ is the slope of the tangent to $f(x)$, and $h(x)$ is the slope of the tangent to $g(x)$.

Effective Procedures

Peer response. Early in the calculus course we conducted some short writing exercises in class. For one, we instructed the students to write a paragraph on the relationship between lines in the plane and linear equations. The papers were then permuted among the students, and some were read aloud by people other than the authors. We devoted 15 to 20 minutes to discussion of structure and content of the paragraphs that had been read, starting with whether the reader or anyone else claimed to understand what the author had attempted to say.

This peer response exercise is a modification of the peer evaluation process used in the University Writing Courses. There it is used to teach students both what to look for when they attempt improvements of their own writing and to relieve the instructor of the burden of being the sole responder to the writer's prose. Peer response was used formally in the calculus course for a short time only. After that, students were expected to have at least one other person read their work and offer comments, which might then be incorporated into the draft submitted to the instructor.

Shortly after the first lab reports were turned in, we devoted an entire class period to a peer response exercise. We copied all of the reports and distributed them to each of the students. We then directed attention to particular sentences, paragraphs, or sections of the reports and raised questions similar to those raised in the Writing Course, for example:

> What's *happening* here? How many of the verbs in this paragraph announce the action of their sentence? If they do not, what does? Can you turn it into a verb and restructure the sentence accordingly?
>
> Whose story is this? Who is the agent?
>
> What seems to require emphasis within a given sentence? Is it in the stress position? Is there more than one emphasizable point in the sentence? Is there less than one?
>
> What is this paragraph about? Underline the point sentence. What does it mean if we can't agree on the point sentence?

Prior to this session, our attempts to tie the writing requirements to the standards of the Writing Course meant almost nothing to the students, mostly because they had completed only two weeks in both courses. However, we could see the lights go on as they got into a spirited discussion of each other's writing, both good and bad. From this point on, in our perception, the mathematics course and the writing course entered into a symbiotic relationship.

The assignment for the next class was for them to mark up the copies of all the lab reports with suggestions for improvements. These marked copies were then distributed to the authors for use in preparing the second submission of the assignment.

Double submission. Another especially effective technique for coaxing good written work out of students is double submission. The students rewrite their papers after the first response from the instructor, with only the second version being assigned a grade. This effectively conveys the idea that rewriting is a *normal* activity for a writer. Moreover, it provides a built-in "second chance," a very popular feature with students.

Late in the semester, when the time frame no longer permitted double submission, we told the students to do their own evaluations on what they had considered their final draft and to revise it accordingly before turning it in. Most of them were so successful in incorporating these new revision techniques into their initial writing process that the quality of these late-term first drafts equalled the quality of early-term second drafts. We believe this was due to the students having experienced, perhaps for the first time, *real* revision. Prior to this experience, their so-called second drafts of papers had mostly been limited to cosmetic firming up and smoothing over—a word in here and out of there, a misspelling respelled, a grammatical error corrected. They had learned to replace *correction* with *re-vision*.

Efficacious instructor responses. To get substantial improvement on the second draft, instructors must attend not only to the correctness of the written product but also to the nature of the process by which it was written. They must focus their comments less on factual correctness and more on whether the student's prose style is capable of producing reasoned and cohesive thought. Mathematical errors must be indicated, of course, and mistakes in grammar, punctuation, and spelling may be; but the student should not be allowed the impression that correction of these mistakes is all that is needed for the second submission. In particular, the mathematical errors are often intimately related to the student's intellectual process or to essential gaps in that process. The most helpful comments for the student are those that highlight the connections between structure and content. The reader expectation methodology is perfectly geared to attend to that.

To make effective use of the methodology, an instructor need not read every sentence or every paragraph of a submission. Selective reading of one paragraph from the beginning, the middle, and the end of the paper will be sufficient to discover whether this particular student is one who consistently foils a particular reader expectation. Once that is discovered, one returns the paper to the student, explains how to go about restructuring the prose through revising the style, and waits for a greatly improved second submission to appear. The double submission procedure, counterintuitive though this might seem, actually takes *less* time than single submission. The time and energy expended in reading the whole of the earlier, less communicative draft will tend to be greater than the time and energy needed to critique two or three paragraphs of the first and glide through the greatly improved prose of the second.

Students are quick to learn that if their first submissions are not riddled with structural problems, the instructor will quickly ascertain *what* it was they were trying to say and can then attend to the *substance* of that clearly articulated thought. This in turn allows students to produce a second draft that does not merely *clarify* the earlier draft, but rather goes *beyond* it. They quickly perceive that it is in their best interest to make the paper readable on the first pass.

To illustrate how much good these procedures seem to have done in our first attempt at this, we give an example from a final examination paper. (We are therefore dealing with an in-class written response to a question not previously encountered.) The problem was to determine the temperature at which the density of water is greatest, given a cubic polynomial formula from the *Handbook of Chemistry and Physics* for the *volume* of a given mass of water as a function of temperature between 0 and 30 degrees Celsius. The student has correctly explained that density will be maximized when volume is minimized. She has also calculated the derivative of the volume function and used the quadratic formula to find its roots at $T = 3.9597$ and $T = 79.5946$. She continues:

> Immediately, we can disregard $T = 79.5946$ because it doesn't lie in the domain. $T = 3.9597$ is the value for T when $V'(T) = 0$ in $[0, 30]$.
>
> However, maximums also occur when $V'(T) = 0$. A minimum will occur when the slope changes from negative to positive. To verify that $T = 3.9597$ is a minimum, we must find the slopes at two points on either side of this point.

[Calculation of $V'(3) = -1.5 \times 10^{-5}$ and $V'(4) = 6.2 \times 10^{-7}$ follows.]

> Since the slope changes from negative to positive at $T = 3.9597$, we can conclude that $T = 3.9597$ is a minimum.
>
> Therefore, the maximum density of water occurs at 3.9597 degrees Celsius. This answer is consistent with what I have learned about water's density in the past—that its maximum density occurs at approximately 4 degrees Celsius.

We can still see room for improvement both in the writing and in the strategy for solving the problem; but for a first-semester freshman with no prior exposure to calculus, this is quite a sophisticated statement about differential calculus and its connections to the real world.

It would not be accurate to say there were no further problems with writing in this course, but the overnight transformation in what the students found they could do was really remarkable. In our responses to their subsequent first drafts, we needed to spend less time on matters of structure, thereby freeing us to concentrate more on substance. For example, students still had to come to grips with the nature of a laboratory experiment and of a report on that experiment, even though these should have been familiar to them from some lab science course. Since their mathematical training, not surprisingly, emphasized reporting only "answers" that had to be "correct," they found it difficult to record and report "data" and to draw inferences from that data. Some of the content problems revealed fundamental misunderstandings of mathematical matters, which they found painful to reveal. Their past experience had enabled them to finesse lack of understanding by patterning a computation after some example in the text, which would usually lead to a "correct" answer, except for careless "math" errors, which were often treated lightly, resulting in the mathematical grace known as "partial credit."

But with all the qualifications, we feel that this initial experiment has proved an exciting success. Several things are clear to us:

1) Thought and expression of thought are so inextricably intertwined for students that improving one will improve the other;

2) In particular, writing assignments in mathematics courses will improve student comprehension;

3) Mathematics teachers can incorporate writing assignments into their courses with significant success and without unduly burdensome extra effort;

4) Reader expectation theory succeeds in producing not only better writing, but also better writers.

Perhaps most remarkable of all, the students, by their course evaluations, graded us A for having added this writing component to their calculus course. Quoth one: "Putting theory into words often was—and still is—a challenge; but this helped me really *learn*, not just memorize, the concepts." Quoth another (an electrical engineering major): "One day I might actually have to *use* these writing skills."

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The Silliness of Reality...

Numbers as realities misbehave. However, there is an ancient and innate sense in people that numbers ought not to misbehave. There is something clean and pure in the abstract notion of number, removed from counting beads, dialects, or clouds; and there ought to be a way of talking about numbers without always having the silliness of reality come in and intrude.

Douglas R. Hofstadter,
Godel, *Escher*, *Bach*: *An Eternal Golden Braid*, Vintage